

Optimal switching problem and system of reflected multi-dimensional FBSDEs with random terminal time

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Abstract

In this paper, we study the solvability of a class of multi-dimensional forward backward stochastic differential equations (FBSDEs) with oblique reflection and unbounded stopping time. Under some mild assumptions on the coefficients in such FBSDE, the existence result of adapted solutions is done via a penalization method. The uniqueness is obtained by a verification theorem similarly to the one used by Hu and Tang [7]. Finally, we establish the connection with the corresponding optimal switching problem. This latter is solved by using the previous results on FBSDEs.

Key Words. Backward stochastic differential equations; Oblique reflection; Optimal switching; Unbounded stopping time; Switching problem.

AMS Subject Classifications. 60H10, 93E20

1 Introduction

This paper is dedicated to the study of a system of multi-dimensional reflected forward-backward stochastic differential equations (FBSDEs in short) with stopping time not necessarily bounded. In fact, we generalize the work of Hu and Tang [7] to infinite horizon.

For $i \in \Lambda := \{1, \dots, d\}$ and $t \geq 0$, we define the forward stochastic differential equation (SDE) by

$$X_i(t) = x_0 + \int_0^{t \wedge \tau} b(s, X_i(s), i) ds + \int_0^{t \wedge \tau} \sigma(s, X_i(s), i) dW_s, \quad (1.1)$$

and the oblique reflected multi-dimensional backward stochastic differential equation (RBSDE) by

$$\begin{cases} Y_i(t) = g(X_i(\tau)) + \int_{t \wedge \tau}^{\tau} f(s, X_i(s), Y_i(s), Z_i(s), i) ds + \int_{t \wedge \tau}^{\tau} dK_i(s) - \int_{t \wedge \tau}^{\tau} Z_i(s) dW(s), \\ Y_i(t) \geq \max_{j \in \mathcal{I}} \{Y_j(t) - C_{i,j}\}, \\ \int_0^{\tau} \left(Y_i(s) - \max_{j \neq i} \{Y_j(s) - C_{i,j}\} \right) dK_i(s) = 0. \end{cases} \quad (1.2)$$

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RBSDEs were firstly studied by El Karoui et al. [5] for the one dimensional case. Later Gegout-Petit and Pardoux [8] extended this work to the multi-dimensional case with reflection on a boundary convex domain, and recently Hu and Tang [7] studied the case of RBSDEs with oblique reflection. In the case of unbounded stopping time, Pardoux [10] gave existence and uniqueness results of BSDEs under one kind of Lipschitz and monotone assumptions. In the infinite horizon, Hamadène et al. [6], Akdim and Ouknine [1] studied reflected BSDEs and reflected BSDEs with jumps respectively. However, for multi-dimensional reflected FBSDEs we find only the work of El Asri [4], in which the author studied a system of reflected FBSDE and provided an application to optimal switching problem, but this work suffers from two points: *i)* The generator depends only on the forward process. *ii)* The infinite horizon value of the solution must be zero.

The novelty of this paper lies in the fact that the generator of the BSDE with stopping time depends on the solution Y_i and the process Z_i . Here the stopping time is unbounded. When the stopping time takes infinity, the value of the solution for FBSDE is not necessarily required to be zero. We then prove existence and uniqueness of the solution under one kind of Lipschitz and monotone assumptions. This kind of stopping time will be used to deal with a switching control problem. Given a switching strategy $\alpha \in \mathcal{A}$, with \mathcal{A} the set of admissible strategies, associated to the controlled process X^α and defined by

$$\alpha_t := \sum_{k \geq 0} \zeta_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t), \quad t \geq 0,$$

here, τ_k with $k \in \mathbb{R}^+$ is a stopping time such that $\lim_{k \rightarrow \infty} \tau_k = \tau$ and ζ_k is an \mathcal{F}_{τ_k} -measurable variable with values in Λ . We consider the total profit at horizon τ defined by

$$J(\alpha) = \mathbb{E}^\alpha \left[g(X_\tau^\alpha) + \int_0^\tau l(s, X^\alpha(s), \alpha_s) ds + \sum_{i \geq 1} C_{\alpha_{i-1}, \alpha_i} \right],$$

where \mathbb{E}^α is the expectation under probability P^α defined in (5.2). The optimal switching problem is to maximize the profit $J(\alpha)$ with respect to α , i.e., find an optimal strategy α^* such that

$$J(\alpha^*) = \sup_{\alpha \in \mathcal{A}} J(\alpha).$$

More details on the practical implications of this type of optimal switching problem are given in [2] and [11].

This paper is organized as follows. In Section 2, we state some assumptions and we discuss the case of X_τ with τ takes infinity. In Section 3 we prove the existence by a penalization method under one kind of Lipschitz and monotone assumptions, whereas in Section 4 we study the uniqueness via a verification theorem. The last section is devoted to the link between the reflected FBSDEs and the optimal switching problem.

Notations. Throughout this paper, we are given a final time τ which is an \mathcal{F} -stopping time not necessarily bounded and a probability space (Ω, \mathcal{F}, P) endowed with a d dimensional Brownian motion $W = (W_t)_{t \geq 0}$. $\{\mathcal{F}_t, t \geq 0\}$ is the natural filtration of the Brownian motion augmented by P -null sets of \mathcal{F} . All the measurability notion will refer to this filtration. We denote by:

S^2 the set of \mathbb{R}^d -valued adapted and càdlàg processes $\{Y(t)\}_{t \geq 0}$ such that

$$\|Y\|_{S^2} := \mathbb{E} \left[\sup_{0 \leq t \leq \tau} |Y(t)|^2 \right]^{1/2} < +\infty.$$

M^2 denotes the set of predictable processes $\{Z(t)\}_{t \geq 0}$ with values in $\mathbb{R}^{d \times p}$ such that

$$\|Z\|_{M^2} := \mathbb{E} \left[\int_0^\tau |Z(s)|^2 ds \right]^{1/2} < +\infty.$$

A^2 is the closed subset of S^2 consisting of nondecreasing processes $K = (K_t)_{0 \leq t \leq \tau}$ with $K_0 = 0$.
 Q the set of process $(y_1, \dots, y_d)^T \in \mathbb{R}^d$ such that

$$y_i > y_j - C_{i,j}, \quad \forall i, j \in \Lambda \text{ s.t } i \neq j,$$

where C is a real function defined on $\Lambda \times \Lambda$.

\bar{Q} is the closer of domain Q in which the reflected BSDE (1.2) evolves, this closer domain is convex and unbounded.

As explained in Hu and Tang [7], each equation of (1.2) is independent of others in the interior of \bar{Q} and on boundary ∂Q of domain Q defined by $\partial Q = \cup_{k=1}^d \partial L_k^+$, where for any $k \in \Lambda$

$$\partial L_k^+ := \{y \in \mathbb{R}^d : y_k > y_l - C_{k,l}, \text{ for any } l \in \Lambda \text{ such that } k \neq l\},$$

the k -th equation is switched to another one, and the solution is reflected along the oblique direction e_k which is positive direction of k -th coordinate axis.

2 Preliminaries

Let us introduce some notations, throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the usual scalar product and the Euclidean norm for vectors respectively, and by $\|\cdot\|$ the trace norm for the matrices. Now, we make the following assumptions:

(H1) The functions $b : \mathbb{R}^+ \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : \mathbb{R}^+ \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ and $f : \mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \times \Lambda \rightarrow \mathbb{R}$. Moreover, $b(\cdot, x)$, $\sigma(\cdot, x)$, $g(x)$ and $f(\cdot, x, y, z, i)$ are all progressively measurable for each $(x, y, z, i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p \times \Lambda$.

(H2) $f(\cdot, 0, 0, 0) := (f(\cdot, 0, 0, 0, 1), \dots, f(\cdot, 0, 0, 0, d))^T$ belongs to M^2 .

(H3) For any $t \geq 0$, $x, x', y, y', z \in \mathbb{R}$ and $i \in \Lambda$ there exist $\mu_1, \mu_2 \in \mathbb{R}$, $\mu_3 \in \mathbb{R}^+$ and one positive deterministic bounded function $u(t)$, such that

$$\langle x - x', b(t, x, i) - b(t, x', i) \rangle \leq \mu_1 |x - x'|^2, \quad \mathbb{P} - \text{a.s.}, \quad (2.1)$$

$$\langle y - y', f(t, x, y, z, i) - f(t, x, y', z, i) \rangle \leq \mu_2 u(t) |y - y'|^2, \quad \mathbb{P} - \text{a.s.}, \quad (2.2)$$

and $\int_0^\infty u(t) dt < \infty$, $\int_0^\infty u^2(t) dt < \infty$.

(H4) For any t, x, x', y, y', z, z' there exist $k \geq 0$ such that

$$|b(t, x, i) - b(t, x', i)| + |\sigma(t, x, i) - \sigma(t, x', i)| \leq u(t) |x - x'|, \quad (2.3)$$

$$|f(t, x, y, z, i) - f(t, x', y', z', i)| \leq u(t) (|x - x'| + \|y - y'\| + \|z - z'\|). \quad (2.4)$$

(H5) For any t, x, x' there exist $k_2 \geq 0$ such that

$$|g(x) - g(x')| \leq k_2 |x - x'|. \quad (2.5)$$

(H6) There exist a constant $\lambda \in \mathbb{R}$ such that for any $i \in \Lambda$, a positive constant C_u depending on the function u , and $\rho, \varepsilon > 0$

$$\begin{aligned} \varepsilon^{-1} C_u + 2\mu_2 u(t) + 2\rho^{-1} u^2(t) + 2\varepsilon &< \lambda < -2\mu_1 - u(t), \quad t \geq 0, \\ \mathbb{E} \left(\int_0^\tau e^{\lambda t} (|b(t, 0, i)|^2 + \|\sigma(t, 0, i)\|^2) dt \right) &< \infty, \\ \mathbb{E} \left(e^{\lambda \tau} |g(0)|^2 + \int_0^\tau e^{\lambda t} |f(t, 0, 0, 0, i)|^2 dt \right) &< \infty. \end{aligned} \quad (2.6)$$

(H7) For any $i \in \Lambda$ and $\tau \in [0, +\infty]$ we have

$$\mathbb{E} \left(\int_0^\tau |b(s, 0, i)| ds \right)^2 + \mathbb{E} \int_0^\tau |\sigma(s, 0, i)|^2 ds < \infty. \quad (2.7)$$

Remark 2.1. For simplicity, we take the same function $u(t)$ in (2.2), (2.4) and (2.6).

The reflected BSDE (1.2) evolves in the closure \bar{Q} of domain Q . As a preparation, we first recall a lemma which is proved by Yin [14]:

Lemma 2.1. (See [14, Remark 2.1 and Lemma 3.2])

Assume (2.1), (2.3) and (2.6) hold, where $\lambda < -2\mu_1 - k_1$. Then the forward SDE (1.1) admits a unique solution $\{X(t)\}_{t \geq 0}$ satisfying

$$\mathbb{E} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} |X(t)|^2 + \int_0^\tau e^{\lambda t} |X(t)|^2 dt \right) < \infty.$$

Before proving existence, we shall discuss the case of X_τ with $\{\tau = +\infty\}$ which appears in the BSDE (1.1). Under Hypothesis (2.3) and (2.7), the integral $\int_0^\infty \sigma(s, X_i(s), i) dW_s$ is well defined and is an L^2 -bounded martingale. Thus, it is easy to show that

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\left| \int_0^t \sigma(s, X_i(s), i) dW_s - \int_0^\infty \sigma(s, X_i(s), i) dW_s \right|^2 \right] = 0.$$

Now, we define

$$\mathcal{X} = x_0 + \int_0^\infty b(s, X_i(s), i) ds + \int_0^\infty \sigma(s, X_i(s), i) dW_s.$$

Then from (1.1) we have

$$\mathcal{X} - X_\tau = \int_\tau^\infty b(s, X_i(s), i) ds + \int_\tau^\infty \sigma(s, X_i(s), i) dW_s, \quad \forall t \geq 0.$$

It is obvious that $\lim_{\tau \rightarrow \infty} \mathbb{E} |\mathcal{X} - X_\tau|^2 = 0$, so that $\mathcal{X} = \lim_{\tau \rightarrow \infty} X_\tau$ in L^2 and we denote it by X_∞ . For more details on the process X_τ with $\tau \in [0, \infty]$, we send the reader to [13].

3 Existence

In this section, we shall prove an existence theorem of solution of FBSDE (1.1)-(1.2). Our setup contains the case $\{\tau \equiv +\infty\}$ as a particular case. Let us firstly make the following assumptions on the cost function C which are standard in the optimal switching problem.

Hypothesis 3.1.

(i) For any $(i, j) \in \Lambda \times \Lambda$, $C_{i,j} \geq 0$.

(ii) For any $(i, j, l) \in \Lambda \times \Lambda \times \Lambda$, such that $i \neq j$ and $j \neq l$, we have

$$C_{i,j} + C_{j,l} \geq C_{i,l}.$$

For $n \geq 0$, let us introduce the following penalized BSDE for any $t \geq 0$ and $i \in \Lambda$:

$$\begin{aligned} Y_i^n(t) &= g(X_i(\tau)) + \int_{t \wedge \tau}^{\tau} f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) - \int_{t \wedge \tau}^{\tau} Z_i^n(s) dW(s) \\ &\quad + n \sum_{l=1}^d \int_{t \wedge \tau}^{\tau} (Y_i^n(s) - Y_l^n(s) + C_{i,l})^- ds. \end{aligned} \quad (3.1)$$

Define

$$K_t^n = n \sum_{l=1}^d \int_0^{t \wedge \tau} (Y_i^n(s) - Y_l^n(s) + C_{i,l})^- ds$$

Note that when $l = i$, we have

$$(Y_i^n(s) - Y_l^n(s) + C_{i,l})^- = 0. \quad (3.2)$$

From the classical result of Chen [3], for any $n \geq 0$, BSDE (3.1) has a unique solution (Y^n, Z^n) in the space $S^2 \times M^2$.

We're going to prove that the triplet (Y^n, Z^n, K^n) converges to the solution of RBSDE (1.2). To do so, we first need the following a priori estimation.

3.1 A priori estimation

In this subsection, we derive two lemmas on the a priori estimation of the penalized BSDE (3.1), which will play a primordial role in the sequence (Y^n, Z^n, K^n) convergence proof.

Lemma 3.1. *Let the Hypotheses (H2), (2.2), (2.4) and (2.6) hold true and assume that $\forall i \in \Lambda$, $g(X_i(\tau)) \in L^2(\Omega, \mathcal{F}_\tau, P, \mathbb{R}^d)$ takes values in \bar{Q} . Then there exists a constant $C > 0$ (independent of n), such that*

$$\begin{aligned} &\mathbb{E} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} \left| (Y_i^n(t) - Y_j^n(t) + C_{i,j})^- \right|^2 + n^2 \int_0^\tau e^{\lambda t} \left| (Y_i^n(t) - Y_j^n(t) + C_{i,j})^- \right|^2 dt \right) \\ &\leq C_u \mathbb{E} \left(\int_0^\tau e^{\lambda s} (|f(s, 0, 0, 0)|^2 + |X_i(s)|^2 + |X_i(s) - X_j(s)|^2) + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 ds \right), \end{aligned} \quad (3.3)$$

where C_u is a constant depending on $u(t)$.

Proof.

For simplicity, denote $\bar{Y}_{ij}^n = Y_i^n(t) - Y_j^n(t) + C_{i,j}$, we have for all $t \geq 0$, $i \in \Lambda$

$$\begin{aligned} \bar{Y}_{ij}^n(t) &= \bar{Y}_{ij}^n(T) + \int_{t \wedge \tau}^{\tau} [f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) - f(s, X_j(s), Y_j^n(s), Z_j^n(s), j)] ds \\ &\quad + n \sum_{l=1}^d \int_{t \wedge \tau}^{\tau} \bar{Y}_{il}^n(s)^- ds - n \sum_{l=1}^d \int_{t \wedge \tau}^{\tau} \bar{Y}_{jl}^n(s)^- - \int_{t \wedge \tau}^{\tau} [Z_i^n(s) - Z_j^n(s)] dW(s). \end{aligned} \quad (3.4)$$

For $i, j \in \Lambda$, if we denote L_{ij}^n the local time of the semi-martingale $\bar{Y}_{ij}^n(t)$, then we get by Tanaka formula

$$\begin{aligned} &\bar{Y}_{ij}^n(t)^- + n \sum_{l=1}^d \int_{t \wedge \tau}^{\tau} I_{\mathcal{L}_{ij,n}}(s) [\bar{Y}_{il}^n(s)^- - \bar{Y}_{jl}^n(s)^-] ds + \frac{1}{2} \int_{t \wedge \tau}^{\tau} dL_{ij}^n(s) \\ &= \int_{t \wedge \tau}^{\tau} I_{\mathcal{L}_{ij,n}}(s) [f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) - f(s, X_j(s), Y_j^n(s), Z_j^n(s), j)] ds \\ &\quad - \int_{t \wedge \tau}^{\tau} I_{\mathcal{L}_{ij,n}}(s) [Z_i^n(s) - Z_j^n(s)] dW(s), \end{aligned}$$

where for $i, j \in \Lambda$,

$$\mathcal{L}_{ij,n} := \{(s, \omega) : \bar{Y}_{ij}^n(s) < 0\}. \quad (3.5)$$

Applying Itô's formula for $e^{\lambda t \wedge \tau} |\bar{Y}_{ij}^n(t)^-|^2$ yields

$$\begin{aligned} & e^{\lambda t} |\bar{Y}_{ij}^n(t)^-|^2 + (2n + \lambda) \int_{t \wedge \tau}^{\tau} e^{\lambda s} |\bar{Y}_{ij}^n(s)^-|^2 ds + \int_{t \wedge \tau}^{\tau} I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} |Z_i^n(s) - Z_j^n(s)|^2 ds \\ = & 2 \int_{t \wedge \tau}^{\tau} I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} \bar{Y}_{ij}^n(s)^- [f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) - f(s, X_j(s), Y_j^n(s), Z_j^n(s), j)] ds \\ & - 2 \int_{t \wedge \tau}^{\tau} I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} \bar{Y}_{ij}^n(s)^- [Z_i^n(s) - Z_j^n(s)] dW(s) \\ & + 2n \int_{t \wedge \tau}^{\tau} e^{\lambda s} \bar{Y}_{ij}^n(s)^- \bar{Y}_{ji}^n(s)^- ds + 2n \sum_{l \neq i, l \neq j} \int_{t \wedge \tau}^{\tau} e^{\lambda s} \bar{Y}_{ij}^n(s)^- [\bar{Y}_{jl}^n(s)^- - \bar{Y}_{il}^n(s)^-] ds, \end{aligned} \quad (3.6)$$

since we have

$$\int_{t \wedge \tau}^{\tau} \bar{Y}_{ij}^n(s)^- dL_{ij}^n(s) = 0, \quad \forall t \geq 0.$$

From other side, since $C_{i,j} + C_{j,i} \geq 0$, then $\bar{Y}_{ji}^n(s)^- \bar{Y}_{ij}^n(s)^- = 0$. In fact

$$\{y \in \mathbb{R}^d : y - y' + C_{i,j} < 0\} \cap \{y \in \mathbb{R}^d : y' - y + C_{j,i} < 0\} = \emptyset.$$

Also we know that for two real numbers x_1 and x_2 , we have $x_1^- - x_2^- \leq (x_1 - x_2)^-$, then

$$\begin{aligned} I_{\mathcal{L}_{ij,n}}(s) [\bar{Y}_{jl}^n(s)^- - \bar{Y}_{il}^n(s)^-] & \leq I_{\mathcal{L}_{ij,n}}(s) [\bar{Y}_{jl}^n(s) - \bar{Y}_{il}^n(s)]^- \\ & = I_{\mathcal{L}_{ij,n}}(s) (Y_j^n(s) - Y_i^n(s) + C_{j,l} - C_{i,l})^- = 0, \end{aligned}$$

in view that

$$\{y \in \mathbb{R}^d : y - y' + C_{i,j} < 0\} \cap \{y \in \mathbb{R}^d : y' - y + C_{j,l} - C_{i,l} < 0\} = \emptyset.$$

Combining this together with (3.6) and taking expectation, we get

$$\begin{aligned} & \mathbb{E} \left(e^{\lambda t \wedge \tau} |\bar{Y}_{ij}^n(t)^-|^2 \right) + (2n + \lambda) \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} |\bar{Y}_{ij}^n(s)^-|^2 ds \\ & + \mathbb{E} \int_{t \wedge \tau}^{\tau} I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} |Z_i^n(s) - Z_j^n(s)|^2 ds \\ \leq & 2 \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} \bar{Y}_{ij}^n(s)^- |f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) - f(s, X_j(s), Y_j^n(s), Z_j^n(s), j)| ds \\ \leq & 2 \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} \bar{Y}_{ij}^n(s)^- \left[|f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) - f(s, X_i(s), Y_i^n(s), Z_i^n(s), j)| \right. \\ & \left. + |f(s, X_i(s), Y_i^n(s), Z_i^n(s), j) - f(s, X_j(s), Y_j^n(s), Z_j^n(s), j)| \right] ds \\ \leq & 2 \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} u(s) \bar{Y}_{ij}^n(s)^- \left[u^{-1}(s) f(s, 0, 0, 0) + |X_i(s)| + |Y_i^n(s)| + |Z_i^n(s)| \right. \\ & \left. + |X_i(s) - X_j(s)| + |\bar{Y}_{ij}^n(s)| + |Z_i^n(s) - Z_j^n(s)| \right] ds \\ \leq & \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} (1 + u(s) + 5u^2(s)) |\bar{Y}_{ij}^n(s)^-|^2 ds \\ & + \frac{1}{2} \mathbb{E} \int_{t \wedge \tau}^{\tau} I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} \left[|f(s, 0, 0, 0)|^2 + |X(s)|^2 + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 \right. \\ & \left. + |X_i(s) - X_j(s)| + |Z_i^n(s) - Z_j^n(s)|^2 \right] ds. \end{aligned} \quad (3.7)$$

Applying Gronwall's inequality and Lemma 2.1, it follows that

$$\begin{aligned} & \mathbb{E} \left(e^{\lambda t \wedge \tau} |\bar{Y}_{ij}^n(t)^-|^2 \right) \\ & \leq C_u \mathbb{E} \int_0^\tau I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} \left[|f(s, 0, 0, 0)|^2 + |X_i(s)|^2 + |X_i(s) - X_j(s)|^2 + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 \right] ds, \end{aligned}$$

and

$$\begin{aligned} & n \mathbb{E} \int_{t \wedge \tau}^\tau I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} |\bar{Y}_{ij}^n(s)^-|^2 ds + \mathbb{E} \int_{t \wedge \tau}^\tau I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} |Z_i^n(s) - Z_j^n(s)|^2 ds \\ & \leq C_u \mathbb{E} \int_0^\tau I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} \left[|f(s, 0, 0, 0)|^2 + |X_i(s)|^2 + |X_i(s) - X_j(s)|^2 + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 \right] ds, \end{aligned}$$

where C_u is a constant depending on $u(t)$ and that will play a crucial role in Lemma 3.2 below. It then follows from Burkholder-Davis-Gundy's inequality applied to (3.6)

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq \tau} e^{\lambda t} |\bar{Y}_{ij}^n(t)^-|^2 \right] \\ & \leq C_u \mathbb{E} \int_0^\tau I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} \left[|f(s, 0, 0, 0)|^2 + |X_i(s)|^2 + |X_i(s) - X_j(s)|^2 + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 \right] ds. \end{aligned}$$

Now from the first inequality in (3.7), we get

$$\begin{aligned} & (2n + \lambda) \mathbb{E} \int_0^\tau I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} |\bar{Y}_{ij}^n(s)^-|^2 ds \\ & \leq \left(n + \frac{C_u}{n} \right) \mathbb{E} \int_0^\tau e^{\lambda s} |\bar{Y}_{ij}^n(s)^-|^2 ds \\ & \quad + \frac{C_u}{n} \mathbb{E} \int_0^\tau I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} \left[|f(s, 0, 0, 0)|^2 + |X_i(s)|^2 + |X_i(s) - X_j(s)|^2 + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 \right] ds. \end{aligned}$$

For n large enough we finally deduce that

$$\begin{aligned} & n^2 \mathbb{E} \int_0^\tau e^{\lambda s} |\bar{Y}_{ij}^n(s)^-|^2 ds \\ & \leq C_u \mathbb{E} \int_0^\tau I_{\mathcal{L}_{ij,n}}(s) e^{\lambda s} \left[|f(s, 0, 0, 0)|^2 + |X_i(s)|^2 + |X_i(s) - X_j(s)|^2 + |Y_i^n(s)|^2 + |Z_i^n(s)|^2 \right] ds. \end{aligned}$$

□

Then, we are able to prove the following estimation:

Lemma 3.2. Assume (H2), (2.2), (2.4), (2.5) and (2.6) hold true. Let us also assume that $\forall i \in \Lambda$, $g(X_i(\tau)) \in L^2(\Omega, \mathcal{F}_\tau, P; \mathbb{R}^d)$ takes values in \bar{Q} . Then there exists a constant $C > 0$, such that

$$\begin{aligned} & \mathbb{E} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_i^n(t)^-|^2 + \int_0^\tau e^{\lambda t} |Y_i^n(t)|^2 dt + \int_0^\tau e^{\lambda t} |Z_i^n(t)|^2 dt \right) \\ & \leq C \mathbb{E} \left(e^{\lambda \tau} |g(0)|^2 + \sup_{0 \leq t \leq \tau} e^{\lambda t} |X_i(t)|^2 + \int_0^\tau e^{\lambda s} \left[|f(s, 0, 0, 0)|^2 + |X_i(s)|^2 + \sum_{j=1}^d |X_i(s) - X_j(s)|^2 \right] ds \right), \end{aligned} \quad (3.8)$$

where $\lambda > \varepsilon^{-1} C_u + 2\mu_2 u(t) + 2\rho^{-1} u^2(t) + 2\varepsilon$ and C depends only on k_2 , ε , ρ and the function u .

Proof.

Applying Itô's formula to $e^{\lambda t \wedge \tau} |Y_i^n(t)|^2$, we obtain:

$$\begin{aligned}
& e^{\lambda t} |Y_i^n(t)|^2 + \int_{t \wedge \tau}^{\tau} e^{\lambda s} (\lambda |Y_i^n(s)|^2 + |Z_i^n(s)|^2) ds \\
&= e^{\lambda \tau} |g(X_i(\tau))|^2 + 2 \int_{t \wedge \tau}^{\tau} e^{\lambda s} Y_i^n(s) \cdot \left[f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) + n \sum_{l=1}^d (Y_i^n(s) - Y_l^n(s) + C_{i,l})^- \right] ds \\
&\quad - 2 \int_{t \wedge \tau}^{\tau} e^{\lambda s} Z_i^n(s) dW(s). \tag{3.9}
\end{aligned}$$

Taking expectation and using the fact that for any arbitrary $\varepsilon > 0$ and any $\rho < 1$ arbitrarily close to one,

$$2\langle y, f(t, x, y, z) \rangle \leq (2\mu_2 u(t) + 2\rho^{-1} u^2(t) + \varepsilon) |y|^2 + \rho \|x\|^2 + \rho \|z\|^2 + \varepsilon^{-1} |f(t, \cdot, 0, 0)|^2,$$

combined with Lemma 3.1 we get

$$\begin{aligned}
& e^{\lambda t \wedge \tau} \mathbb{E} |Y_i^n(t)|^2 + \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} (\lambda |Y_i^n(s)|^2 + \rho |Z_i^n(s)|^2) ds \\
&\leq \mathbb{E} \left[e^{\lambda \tau} |g(0)|^2 \right] + k_2 \mathbb{E} \sup_{0 \leq t \leq \tau} e^{\lambda t} |X_i(t)|^2 + \mathbb{E} \int_{t \wedge \tau}^{\tau} (2\mu_2 u(t) + 2\rho^{-1} u^2(t) + 2\varepsilon) e^{\lambda s} |Y_i^n(s)|^2 ds \\
&\quad + \varepsilon^{-1} \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} |f(s, 0, 0, 0)|^2 ds + \rho \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} [|X_i(s)|^2 + |Z_i(s)|^2] ds \\
&\quad + n^2 \varepsilon^{-1} \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} \sum_{l=1}^d ((Y_i^n(s) - Y_l^n(s) + C_{i,l})^-)^2 ds.
\end{aligned}$$

For $\bar{\lambda} := \lambda - \varepsilon^{-1} C_u - 2\mu_2 u(t) - 2\rho^{-1} u^2(t) - 2\varepsilon > 0$ and $\bar{\rho} = 1 - \rho - \varepsilon^{-1} C_u > 0$, we get

$$\begin{aligned}
& e^{\lambda t} \mathbb{E} |Y_i^n(t)|^2 + \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} (\bar{\lambda} |Y_i^n(s)|^2 + \bar{\rho} |Z_i^n(s)|^2) ds \\
&\leq C \mathbb{E} \left(e^{\lambda \tau} |g(0)|^2 + \sup_{0 \leq t \leq \tau} e^{\lambda t} |X_i(t)|^2 + \int_0^{\tau} e^{\lambda s} [|f(s, 0, 0, 0)|^2 + |X_i(s)|^2 + \sum_{j=1}^d |X_i(s) - X_j(s)|^2] ds \right),
\end{aligned}$$

where C depends on k_2, ε, ρ and the function u . Finally, we deduce by an argument already used. This completes the proof. \square

This will allow us to prove the convergence of the sequence (Y^n, Z^n, K^n) .

3.2 Convergence of the sequence (Y^n, Z^n, K^n)

Now, we will prove that $(Y^n, Z^n, K^n)_{n \geq 0}$ is a Cauchy sequence.

Lemma 3.3. *The sequence $\{(Y^n, Z^n)\}_n$ is a Cauchy sequence in the space $S^2 \times M^2$.*

Proof. Denote:

$$\begin{aligned}
Y_i^n(t) - Y_i^m(t) &= \bar{Y}_i^{n,m}(t), \\
Z_i^n(t) - Z_i^m(t) &= \bar{Z}_i^{n,m}(t), \\
\bar{Y}_{ij}^n &= Y_i^n(t) - Y_j^n(t) + C_{i,j}.
\end{aligned}$$

Applying Itô's formula to $e^{\lambda t \wedge \tau} |\bar{Y}_i^{n,m}(t)|^2$, we have for $i \in \Lambda$

$$\begin{aligned}
& \mathbb{E} \left(e^{\lambda t \wedge \tau} |\bar{Y}_i^{n,m}(t)|^2 \right) + \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} (\lambda |\bar{Y}_i^{n,m}(s)|^2 + |\bar{Z}_i^{n,m}(s)|^2) ds \\
&= 2\mathbb{E} \int_0^{\tau} e^{\lambda s} \bar{Y}_i^{n,m}(s) (f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) - f(s, X_i(s), Y_i^m(s), Z_i^m(s), i)) ds \\
&\quad + 2n\mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} \bar{Y}_i^{n,m}(s) \bar{Y}_{ij}^n(s)^- ds - 2m\mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s} \bar{Y}_i^{n,m}(s) \bar{Y}_{ij}^m(s)^- ds \\
&\leq C_\alpha \mathbb{E} \int_0^{\tau} u^2(s) e^{\lambda s} |\bar{Y}_i^{n,m}(s)|^2 ds + \alpha \mathbb{E} \int_0^{\tau} e^{\lambda s} |\bar{Z}_i^{n,m}(s)|^2 ds \\
&\quad + 2\mathbb{E} \int_{t \wedge \tau}^{\tau} n^2 e^{\lambda s} |\bar{Y}_{ij}^n(s)^-|^2 ds + m^2 e^{\lambda s} |\bar{Y}_{ij}^m(s)^-|^2 ds.
\end{aligned} \tag{3.10}$$

From Lemma 3.1, and by applying Gronwall's Lemma for $\alpha < 1$. We obtain

$$\forall m \geq n, \quad \sup_{0 \leq t \leq \tau} \mathbb{E} \left(e^{\lambda t} |\bar{Y}_i^{n,m}(t)|^2 \right) \leq \frac{C}{n}.$$

We deduce also

$$\forall m \geq n, \quad \mathbb{E} \int_0^{\tau} e^{\lambda t} |\bar{Z}_i^{n,m}(t)|^2 dt \leq \frac{C}{n}.$$

We rewrite again Itô's formula for $e^{\lambda t} |\bar{Y}_i^{n,m}(t)|^2$, using then Burkholder-Davis-Gundy's inequality and some argument already used, we obtain for $i \in \Lambda$,

$$\mathbb{E} \sup_{0 \leq t \leq \tau} e^{\lambda t} |\bar{Y}_i^{n,m}(t)|^2 \leq \frac{C}{n}.$$

□

Let us now define the process $Y_t = \lim_{n \rightarrow +\infty} Y_t^n$ in the sense that Y^n converges to Y in S^2 , and $Z_t = \lim_{n \rightarrow +\infty} Z_t^n$ in the sense that Z^n converges to Z in M^2 .

We define also:

$$K_i^n(t) := n \int_0^{t \wedge \tau} \sum_{l=1}^d (Y_i^n(s) - Y_l^n(s) + C_{i,l})^- ds, \quad i \in \Lambda. \tag{3.11}$$

From the expression of BSDE (3.1), we have

$$K_i^n(t) = Y_i^n(t) - Y_i^n(0) + K_i^n(\tau) + \int_0^{t \wedge \tau} f(s, X_i(s), Y_i^n(s), Z_i^n(s), i) ds - \int_0^{t \wedge \tau} Z_i^n(s) dW(s), \quad i \in \Lambda. \tag{3.12}$$

Set

$$K_i(t) := Y_i(t) - Y_i(0) + \int_0^{t \wedge \tau} f(s, X_i(s), Y_i(s), Z_i(s), i) ds - \int_0^{t \wedge \tau} Z_i(s) dW(s), \quad i \in \Lambda. \tag{3.13}$$

Then, we deduce immediately that K^n converges to K in S^2 .

Finally, it remains to show that

$$\int_0^{\tau} \left(Y_i(s) - \max_{j \neq i} [Y_j(s) - C_{i,j}] \right)^+ dK_i^n(s) = 0, \quad i \in \Lambda. \tag{3.14}$$

However, we have from (3.11) that for $i \in \Lambda$

$$\begin{aligned}
& \int_0^\tau \left(Y_i^n(s) - \max_{j \neq i} [Y_j^n(s) - C_{i,j}] \right)^+ dK_i^n(s) \\
&= n \sum_{l=1}^d \int_0^\tau \left(Y_i^n(s) - \max_{j \neq i} [Y_j^n(s) - C_{i,j}] \right)^+ (Y_i^n(s) - Y_l^n(s) + C_{i,l})^- ds,
\end{aligned}$$

which is equal to zero by construction, then as $n \rightarrow \infty$, from [8, Lemma 5.8] we have (3.14). In fact, we have shown the existence of the solution of the reflected BSDEs (1.2):

Theorem 3.1. *Let the Hypotheses (H1 – H7) hold. Assume that $g(X(\tau)) \in L^2(\Omega, \mathcal{F}_\tau, P; \mathbb{R}^d)$ takes values in \bar{Q} . Then RBSDE (1.2) has a solution (Y, Z, K) in $S^2 \times M^2 \times A^2$.*

4 Verification theorem

A switching strategy α consist in a sequence $\alpha := (\tau_k, \zeta_k)_{k \geq 1}$, where $(\tau_k)_{k \geq 1}$ is an increasing sequence of \mathbb{F} -stopping times smaller than τ , and ζ_k are \mathcal{F}_{τ_k} -measurable random variables valued in Λ . For an initial regime i_0 we define an admissible strategy as follows:

$$\alpha_t := \sum_{k \geq 0} \zeta_k 1_{[\tau_k, \tau_{k+1}]}(t), \quad t \geq 0, \quad (4.1)$$

with $\tau_0 = 0$ and $\zeta_0 = i_0$.

We denote by $\mathcal{A}(t)$ the set of admissible strategies starting at time t and $\mathcal{A}_i(t)$ the subset of $\mathcal{A}(t)$ starting at time t from the mode i

$$\mathcal{A}_i(t) := \{\alpha \in \mathcal{A}(t) : \alpha_t = i\}.$$

For any α . we define the process A^α by

$$A^\alpha(s) = \sum_{k \geq 0} C_{\zeta_k, \zeta_{k+1}} 1_{[\tau_k, \tau]}(s). \quad (4.2)$$

Given a strategy $\alpha \in \mathcal{A}$ we define the following BSDE:

$$U(s) = g(X^{\alpha}(\tau)) + A^\alpha(\tau) - A^\alpha(s) + \int_{s \wedge \tau}^\tau \psi(r, U(r), V(r), \alpha(r)) dr - \int_{s \wedge \tau}^\tau V(r) dW(r), \quad s \geq t. \quad (4.3)$$

This BSDE has a solution in $S^2 \times M^2$ denoted (U^α, V^α) , to prove this, it is enough to write for $s \geq t$

$$\begin{aligned}
\tilde{U}(s) &= U(s) + A^\alpha(s), \\
\tilde{V}(s) &= V(s).
\end{aligned}$$

Then we get from (4.3)

$$\tilde{U}(s) = g(X^\alpha(\tau)) + A^\alpha(\tau) + \int_{s \wedge \tau}^\tau \psi(r, \tilde{U}(r) - A^\alpha(r), \tilde{V}(r), \alpha(r)) dr - \int_{s \wedge \tau}^\tau V(r) dW(r), \quad s \geq t.$$

Which has solution from standard arguments. We impose the following stronger assumptions:

Hypothesis 4.1. (i) For any $(i, j) \in \Lambda \times \Lambda$, $C_{i,j} \geq 0$.

(ii) For any $(i, j, l) \in \Lambda \times \Lambda \times \Lambda$ such that $i \neq j$ and $j \neq l$,

$$C_{i,j} + C_{j,l} > C_{i,l}.$$

With the following representation of the solution of BSDE (1.2), we have immediately the uniqueness of the solution.

Theorem 4.1. *Let us suppose that the Hypotheses (H2), (H4) and 4.1 hold. Let us also assume that $g(X(\tau)) \in L^2(\Omega, \mathcal{F}_\tau, P; \mathbb{R}^d)$ takes values in \bar{Q} . Let $(\tilde{Y}, \tilde{Z}, \tilde{K})$ be a solution in (S^2, M^2, K^2) to RBSDE (1.2). Then*

(i) *For any $\alpha(\cdot) \in \mathcal{A}_i(t)$, we have:*

$$\tilde{Y}_i(t) \leq U^{\alpha(\cdot)}(t), \quad P - a.s. \quad (4.4)$$

(ii) *Set $\tau_0^* = t$, $\zeta_0^* = i$ and define the sequence $\{\tau_j^*, \zeta_j^*\}_{j=1}^\infty$ in an inductive way as follows:*

$$\tau_j^* := \inf\{s \geq \tau_{j-1}^* : \tilde{Y}_{\zeta_{j-1}^*}(s) = \max_{l \neq \zeta_{j-1}^*} \{\tilde{Y}_l(s) - C_{\zeta_{j-1}^*, l}\} \wedge \tau, \quad (4.5)$$

and ζ_j^ is $\mathcal{F}_{\tau_j^*}$ -measurable random variable such that*

$$\tilde{Y}_{\zeta_{j-1}^*}(\tau_j^*) = \tilde{Y}_{\zeta_j^*}(\tau_j^*) - C_{\zeta_{j-1}^*, \zeta_j^*},$$

with $j = 1, 2, \dots$.

Then, the following switching strategy:

$$\alpha_s^* = i \mathbb{1}_{\{t\}}(s) + \sum_{j \geq 1} \zeta_{j-1}^* \mathbb{1}_{(\tau_{j-1}^*, \tau_j^*]}(s), \quad (4.6)$$

is admissible, i.e., $\alpha^ \in \mathcal{A}_i(t)$ and we have,*

$$\tilde{Y}_i(t) = U^{\alpha^*(\cdot)}(t).$$

Moreover, $\tilde{Y}(t)$:

$$\tilde{Y}_i(t) = \operatorname{ess\,sup}_{\alpha \in \mathcal{A}_i^t} U^\alpha(t), \quad i \in \Lambda, \quad t \geq 0.$$

Therefore RBSDE (1.2) has a unique solution.

Proof. We prove w.l.o.g (i) and (ii) for the particular case of $t = 0$.

(i) We define

$$\tilde{Y}^{\alpha_\cdot}(s) = \sum_{\substack{i \geq 1 \\ \tau_i \in [0, \tau)}} \tilde{Y}_{\zeta_{i-1}}(s) \mathbb{1}_{[\tau_{i-1}, \tau_i)}(s) + g(X^{\alpha_\tau}(\tau)) \mathbb{1}_{\{\tau\}}(s), \quad (4.7)$$

$$\tilde{Z}^{\alpha_\cdot}(s) = \sum_{\substack{i \geq 1 \\ \tau_i \in [0, \tau)}} \tilde{Z}_{\zeta_{i-1}}(s) \mathbb{1}_{[\tau_{i-1}, \tau_i)}(s), \quad (4.8)$$

$$\tilde{K}^{\alpha_\cdot}(s) = \sum_{\substack{i \geq 1 \\ \tau_i \in [0, \tau)}} \int_{\tau_{i-1} \wedge s}^{\tau_i \wedge s} d\tilde{K}_{\zeta_{i-1}}(r). \quad (4.9)$$

The process $\tilde{Y}^{\alpha_\cdot}(\cdot)$ is càdlàg with jump $\tilde{Y}_{\alpha_i}(\tau_i) - \tilde{Y}_{\alpha_{i-1}}(\tau_i)$ at $\tau_i \in [0, \tau]$, $i \in \Lambda$, it follows that

$$\begin{aligned}
\tilde{Y}^{\alpha \cdot}(s) - \tilde{Y}^{\alpha \cdot}(0) &= \sum_{\substack{i \geq 1 \\ \tau_i \in [0, \tau)}} \int_{\tau_{i-1} \wedge s}^{\tau_i \wedge s} [-f(r, \tilde{Y}_{\zeta_{i-1}}(r), \tilde{Z}_{\zeta_{i-1}}(r), \zeta_{i-1})dr + \tilde{Z}_{\zeta_{i-1}}(r)dW(r) - d\tilde{K}_{\zeta_{i-1}}(r)] \\
&\quad + \sum_{\substack{i \geq 1 \\ \tau_i \in [0, \tau)}} [\tilde{Y}_{\zeta_i}(\tau_i) - \tilde{Y}_{\zeta_{i-1}}(\tau_i)] \mathbb{1}_{[\tau_i, \tau]}(s) \\
&= \int_0^s [-f(r, \tilde{Y}^{\alpha \cdot}(r), \tilde{Z}^{\alpha \cdot}(r), \alpha_r)dr + \tilde{Z}^{\alpha \cdot}(r)dW(r) - d\tilde{K}^{\alpha \cdot}(r)] + \tilde{A}^{\alpha \cdot}(s) - A^{\alpha \cdot}(s),
\end{aligned}$$

where

$$\tilde{A}^{\alpha \cdot}(s) = \sum_{i \geq 1, \tau_i \in [0, \tau)} \left[\tilde{Y}_{\zeta_i}(\tau_i) + C_{\zeta_{i-1}, \zeta_i} - \tilde{Y}_{\zeta_{i-1}}(\tau_i) \right] \mathbb{1}_{[\tau_i, \tau]}(s), \quad (4.10)$$

which is increasing since we have

$$\tilde{Y}(t) \in \bar{Q}, \quad \forall t \geq 0.$$

Thus it implies that $(\tilde{Y}^{\alpha \cdot}, \tilde{Z}^{\alpha \cdot})$ is a solution of the following BSDE:

$$\begin{aligned}
\tilde{Y}^{\alpha \cdot}(s) &= g(X^{\alpha(\tau)}(\tau)) + A^{\alpha \cdot}(\tau) - A^{\alpha \cdot}(s) + [(\tilde{K}^{\alpha \cdot}(\tau) + \tilde{A}^{\alpha \cdot}(\tau)) - (\tilde{K}^{\alpha \cdot}(s) + \tilde{A}^{\alpha \cdot}(s))] \\
&\quad + \int_{s \wedge \tau}^{\tau} f(r, \tilde{Y}^{\alpha \cdot}(r), \tilde{Z}^{\alpha \cdot}(r), \alpha(r))dr - \int_{s \wedge \tau}^{\tau} \tilde{Z}^{\alpha \cdot}(r)dW(r), \quad s \geq 0.
\end{aligned} \quad (4.11)$$

Since both $\tilde{K}^{\alpha \cdot}$ and $\tilde{A}^{\alpha \cdot}$ are increasing càdlàg processes, from the comparison theorem for multi-dimensional infinite horizon BSDEs of Shi and Zhang [12, Theorem 6] we conclude that

$$\tilde{Y}^{\alpha \cdot}(0) \geq U^{\alpha \cdot}(0),$$

which implies that

$$\tilde{Y}_i(0) \geq U^{\alpha \cdot}(0).$$

The rest of the proof is similar to the proof of Theorem 3.1 in [7]. □

5 Application to optimal switching problem with unbounded stopping time

In this section we make the link between the optimal switching problem and the infinite horizon multi-dimensional reflected BSDEs studied previously. We assume that C satisfies Hypothesis 4.1, and we assume also the following hypothesis.

Hypothesis 5.1.

(i) $l(\cdot, 0) := (l(\cdot, 0, 1), \dots, l(\cdot, 0, m))^T$ belongs to M^2 .

(ii) For any t, x, y , and $i \in \Lambda$ there exist $\mu_3 \in \mathbb{R}$ and one positive deterministic bounded function $u(t)$, such that

$$\langle y, l(t, x, i) \rangle \leq \mu_3 |y|^2 + u(t) |x|^2, \quad \mathbb{P} - a.s., \quad (5.1)$$

and $\int_0^\infty u(t)dt < \infty$, $\int_0^\infty u^2(t)dt < \infty$.

(iii) For any t, x and $i \in \Lambda$ we have

$$|l(t, x, i) - l(t, x', i)| \leq u(t) |x - x'|.$$

(iv) σ is invertible and σ^{-1} is bounded.

(v) b is bounded.

Under Hypothesis (H1) and assumptions (iv)-(v), the following stochastic differential equation:

$$dX_t = \sigma(t, X) dW_t, \quad X_0 = x \in \mathbb{R}^d, \quad t \geq 0,$$

has a unique solution. Identically as in the previous section, a switching strategy α consists in a sequence $\alpha := (\tau_k, \zeta_k)_{k \geq 1}$, where $(\tau_k)_{k \geq 1}$ is an increasing sequence of \mathbb{F} -stopping times (i.e. $\tau_0 = 0$, $\tau_k \leq \tau_{k+1}$ and $\lim_{k \rightarrow \infty} \tau_k = \tau$), and ζ_k are \mathcal{F}_{τ_k} -measurable random variables valued in Λ . To a strategy $\alpha = (\tau_k, \zeta_k)_{k \geq 1}$ and an initial regime i_0 , we associate the state process $(\alpha_t)_{t \leq \tau}$ defined by

$$\alpha_t := \sum_{k \geq 0} \zeta_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t), \quad t \geq 0,$$

with $\tau_0 = 0$ and $\zeta_0 = i_0$. We denote \mathcal{A} the set of admissible strategies and \mathcal{A}_i the subset of strategies starting from state $i \in \Lambda$ at time 0:

$$\mathcal{A}_i := \left\{ \alpha \in \mathcal{A} : \alpha_0 = i \text{ and } \mathbb{E}^\alpha \left[\sum_{k \geq 1} C_{\zeta_{k-1}, \zeta_k} \right] < \infty \right\},$$

where \mathbb{E}^α denotes the expectation w.r.t the probability P^α , defined for each $\alpha \in \mathcal{A}_i$ on (Ω, \mathcal{F}) by:

$$\frac{dP^\alpha}{dP} = \exp \left\{ \int_0^\tau b(s, X^\alpha(s), \alpha_s) dW_s - \frac{1}{2} \int_0^\tau |b(s, X^\alpha(s), \alpha_s)|^2 ds \right\}. \quad (5.2)$$

From the assumptions on σ and b , and according to Girsanov's theorem, the process

$$B_t^\alpha = B_t - \int_0^t b(s, X^\alpha(s), \alpha_s) ds, \quad t \geq 0,$$

is a Brownian motion on $(\Omega, \mathcal{F}, P^\alpha)$. Moreover, for each $\alpha \in \mathcal{A}_i$, X^α is a weak solution of:

$$dX_t^\alpha = \sigma(t, X^\alpha(t), \alpha_t) dW_t^\alpha + b(t, X^\alpha(t), \alpha_t) dt, \quad X_0^\alpha = x, \quad t \geq 0. \quad (5.3)$$

Let $(P^\alpha, B^\alpha, X^\alpha)$ be a weak solution of SDE (5.3), associated with the admissible switching strategy $\alpha \in \mathcal{A}_i$. We consider the total profit at horizon τ defined by

$$J(\alpha) = \mathbb{E}^\alpha \left[g(X^\alpha(\tau)) + \int_0^\tau l(s, X^\alpha(s), \alpha_s) ds + \sum_{i \geq 1} C_{\alpha_{i-1}, \alpha_i} \right].$$

The switching problem is to maximize the profit $J(\alpha)$ over $\alpha \in \mathcal{A}_i$, subject to the state equation (5.3), which consists in finding an optimal strategy $\alpha^* \in \mathcal{A}_i$ such that

$$J(\alpha^*) = \sup_{\alpha \in \mathcal{A}_i} J(\alpha).$$

We define f as follows: $\forall (t, x, z, i) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \times \Lambda$,

$$f(t, x, z, i) := l(t, x, i) + \langle z, b(t, x, i) \rangle. \quad (5.4)$$

Under Hypothesis 5.1, and the expression (5.4), the following RBSDE:

$$\begin{cases} Y_i(t) = g(X_i(\tau)) + \int_{t \wedge \tau}^{\tau} f(s, X_i(s), Z_i(s), i) ds + \int_{t \wedge \tau}^{\tau} dK_i(s) - \int_{t \wedge \tau}^{\tau} Z_i(s) dW(s), \\ Y_i(t) \geq \max_{j \in \mathcal{I}} \{Y_j(t) - C_{i,j}(t)\}, \\ \int_0^{\tau} \left(Y_i(s) - \max_{j \neq i} \{Y_j(s) - C_{i,j}(s)\} \right) dK_i(s) = 0, \end{cases} \quad (5.5)$$

has a unique solution $(Y, Z, K) \in S^2 \times M^2 \times A^2$, thanks to Theorems 3.1 and 4.1.

Now we give the main result of this section:

Theorem 5.1. *Let $\alpha^* = (\tau_n^*, \zeta_n^*)_{n \geq 0}$ be the strategy given by $(\tau_0^*, \zeta_0^*) = (0, i_0)$ with $i_0 \in \Lambda$ and defined recursively, for $n \geq 1$, by*

$$\begin{aligned} \tau_n^* &= \inf \left\{ s \geq \tau_{n-1}^*; \quad Y_{\zeta_{n-1}^*}(s) = \max_{j \in \Lambda^{-\zeta_{n-1}^*}} \left(Y_j(s) - C_{\zeta_{n-1}^*, j}(s) \right) \right\} \wedge \tau, \\ \zeta_n^* &\in \operatorname{argmax} \left\{ j; \quad Y_{\zeta_{n-1}^*}(s) = \max_{j \in \Lambda^{-\zeta_{n-1}^*}} \left(Y_j(s) - C_{\zeta_{n-1}^*, j}(s) \right) \right\}, \end{aligned} \quad (5.6)$$

where $\Lambda^{-i} := \Lambda - \{i\}$.

Under Hypotheses 3.1 and 5.1, the strategy α^* is optimal for the switching problem and we have

$$Y_{i_0}(0) = J(\alpha^*) = \sup_{\alpha \in \mathcal{A}_{i_0}} J(\alpha).$$

Proof. The proof is performed in two steps.

Step 1. The strategy α^* satisfies $Y_{i_0}(0) = J(\alpha^*)$.

We consider the reflected BSDE (5.5)

$$Y_{i_0}(t) = g(X_{i_0}(\tau)) + \int_{t \wedge \tau}^{\tau} f(s, X_{i_0}(s), Z_{i_0}(s), i_0) ds + \int_{t \wedge \tau}^{\tau} dK_{i_0}(s) - \int_{t \wedge \tau}^{\tau} Z_{i_0}(s) dW(s).$$

Since $Y_{i_0}(0)$ is deterministic, then

$$\begin{aligned} Y_{i_0}(0) &= \mathbb{E}^{\alpha^*} \left[g(X_{i_0}(\tau)) + \int_0^{\tau} f(s, X_{i_0}(s), Z_{i_0}(s), i_0) ds + \int_0^{\tau} dK_{i_0}(s) - \int_0^{\tau} Z_{i_0}(s) dW(s) \right] \\ &= \mathbb{E}^{\alpha^*} \left[\int_0^{\tau_1^*} f(s, X_{i_0}(s), Z_{i_0}(s), i_0) ds + K_{i_0}(\tau_1^*) - \int_0^{\tau_1^*} Z_{i_0}(s) dW(s) + Y_{i_0}(\tau_1^*) \right] \\ &= \mathbb{E}^{\alpha^*} \left[\int_0^{\tau_1^*} l(s, X_{i_0}(s), i_0) ds + K_{i_0}(\tau_1^*) - \int_0^{\tau_1^*} Z_{i_0}(s) dW^{\alpha^*}(s) + Y_{i_0}(\tau_1^*) \right]. \end{aligned}$$

From the definition of τ_1^* we know that the process $K_{i_0}(\tau_1^*)$ does not increase between 0 and τ_1^* and then $K_{i_0}(\tau_1^*) = 0$. On the other hand using the Burkholder-Davis-Gandy's inequality and the assumptions on b , we have that $\left(\int_0^{t \wedge \tau} Z_{i_0}(s) dW^{\alpha^*}(s), t \geq 0 \right)$ is a P^{α^*} -martingale. Therefore

$$Y_{i_0}(0) = \mathbb{E}^{\alpha^*} \left[\int_0^{\tau_1^*} l(s, X_{i_0}(s), i_0) ds + Y_{i_0}(\tau_1^*) \right].$$

From (5.6), we have $Y_{i_0}(\tau_1^*) = Y_{\zeta_1^*}(\tau_1^*) - C_{i_0, \zeta_1^*}$, therefore

$$Y_{i_0}(0) = \mathbb{E}^{\alpha^*} \left[\int_0^{\tau_1^*} l(s, X_{i_0}(s), i_0) ds + Y_{\zeta_1^*}(\tau_1^*) - C_{i_0, \zeta_1^*} \right].$$

In the same spirit, we repeat this reasoning for $Y_{\zeta_1^*}(\tau_1^*)$. We deduce recursively that

$$Y_{i_0}(0) = \mathbb{E}^{\alpha^*} \left[\sum_{k=1}^n \int_{\tau_{k-1}^*}^{\tau_k^*} l(s, X_{\zeta_k^*}(s), \zeta_k^*) ds + Y_{\zeta_n^*}(\tau_n^*) - \sum_{k=1}^n C_{\zeta_{k-1}^*, \zeta_k^*} \right],$$

where $\sum_{k=1}^n \int_{\tau_{k-1}^*}^{\tau_k^*} l(s, X_{\zeta_k^*}(s), \zeta_k^*) ds = \int_0^{\tau_n^*} l(s, X^{\alpha^*}(s), \alpha^*) ds$.

Then the strategy α^* is admissible i.e $E^{\alpha^*}[\sum_{k \geq 1} C_{\zeta_{k-1}^*, \zeta_k^*}] < +\infty$, because if not, we would have $Y_{i_0}(0) = -\infty$ which contradicts the assumption $Y_{i_0} \in \mathcal{S}^2$. Thus sending n to infinity, we get that

$$Y_{i_0}(0) = \mathbb{E}^{\alpha^*} \left[\int_0^\tau l(s, X^{\alpha^*}(s), \alpha^*) ds - \sum_{k \geq 1} C_{\zeta_{k-1}^*, \zeta_k^*} + Y_{\alpha^*}(\tau) \right],$$

where $Y_{\alpha^*}(\tau) = g(X_\tau)$. Therefore we obtain that $Y_{i_0}(0) = J(\alpha^*)$.

Step 2. The strategy α^* is optimal.

We pick any strategy $\alpha = (\tau_n, \zeta_n)_{n \geq 0} \in \mathcal{A}_{i_0}$, we consider once again the reflected BSDE (5.5)

$$\begin{aligned} Y_{i_0}(0) &= \mathbb{E}^\alpha \left[g(X_{i_0}(\tau)) + \int_0^\tau f(s, X_{i_0}(s), Z_{i_0}(s), i_0) ds + \int_0^\tau dK_{i_0}(s) - \int_0^\tau Z_{i_0}(s) dW(s) \right] \\ &= \mathbb{E}^\alpha \left[\int_0^{\tau_1} f(s, X_{i_0}(s), Z_{i_0}(s), i_0) ds + K_{i_0}(\tau_1) - \int_0^{\tau_1} Z_{i_0}(s) dW(s) + Y_{i_0}(\tau_1) \right] \\ &= \mathbb{E}^\alpha \left[\int_0^{\tau_1} l(s, X_{i_0}(s), i_0) ds + K_{i_0}(\tau_1) - \int_0^{\tau_1} Z_{i_0}(s) dW^{\alpha^*}(s) + Y_{i_0}(\tau_1) \right]. \end{aligned}$$

On the one hand, we know that $Y_{i_0}(\tau_1) \geq Y_{\zeta_1}(\tau_1) - C_{i_0, \zeta_1}$ and $K_{i_0}(\tau_1) \geq 0$. Moreover $\left(\int_0^{t \wedge \tau} Z_{i_0}(s) dW^{\alpha^*}(s), t \geq 0 \right)$ is a P^{α^*} -martingale, therefore:

$$Y_{i_0}(0) \geq \mathbb{E}^\alpha \left[\int_0^{\tau_1} l(s, X_{i_0}(s), i_0) ds + Y_{\zeta_1}(\tau_1) - C_{i_0, \zeta_1} \right].$$

Next we replace $Y_{\zeta_1}(\tau_1)$ by its value using the same reasoning, and by proceeding exactly as in step 1, an induction argument leads to

$$Y_{i_0}(0) \geq \mathbb{E}^\alpha \left[\sum_{k=1}^n \int_{\tau_{k-1}}^{\tau_k} l(s, X_{\zeta_k}(s), \zeta_k) ds + Y_{\zeta_n}(\tau_n) - \sum_{k=1}^n C_{\zeta_{k-1}, \zeta_k} \right].$$

Sending n to infinity, since the strategy is admissible, we get

$$Y_{i_0}(0) \geq \mathbb{E}^\alpha \left[\int_0^\tau l(s, X^\alpha(s), \alpha) ds - \sum_{k \geq 1} C_{\zeta_{k-1}, \zeta_k} + Y^{\alpha_\tau}(\tau) \right],$$

with $Y^{\alpha_\tau}(\tau) = g(X_\tau^{\alpha_\tau})$.

Therefore we obtain that $Y_{i_0}(0) \geq J(\alpha)$. The arbitrariness of α concludes the proof. \square

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